Mechanics

Measuring methods
Determining the gravitational constant

Determining the gravitational constant with the gravitation torsion balance after Cavendish

Measuring the excursion with a light pointer

Objects of the experiment

■ To determine the initial equilibrium position of the torsion pendulum.

■ To record the damped oscillations around the final equilibrium position of the torsion pendulum as a function of time.

■ To determine the final equilibrium position of the torsion pendulum by means of the end-deflection method.

■ To calculate the gravitational constant G from the oscillation time and the difference between the equilibrium positions.

■ To determine the gravitational constant G using the acceleration method.

Principles

The heart of the gravitation torsion balance according to Cavendish is a light transverse beam, horizontally suspended by a thin torsion string, which supports a small lead ball with the mass m_2 at each end at a distance d from the suspension point. These two balls are attracted by two large lead spheres with the mass m_1. Although the force of attraction is less than 10^{-9} N, it is still possible to demonstrate this using the extremely sensitive gravitation torsion balance. The motion of the small lead balls is observed and measured using a light pointer (see Fig. 1).

This light pointer is generated using an illuminated concave mirror which is mounted in a fixed position on the transverse beam of the torsion pendulum. On the basis of the motion over time, the mass m_1 and the geometry of the setup, it is possible to determine the gravitational constant using the end-deflection method or – in an abbreviated measuring procedure – by means of the acceleration method.

Fig. 1: Gravitation torsion balance according to Cavendish, and schematic diagram of light pointer
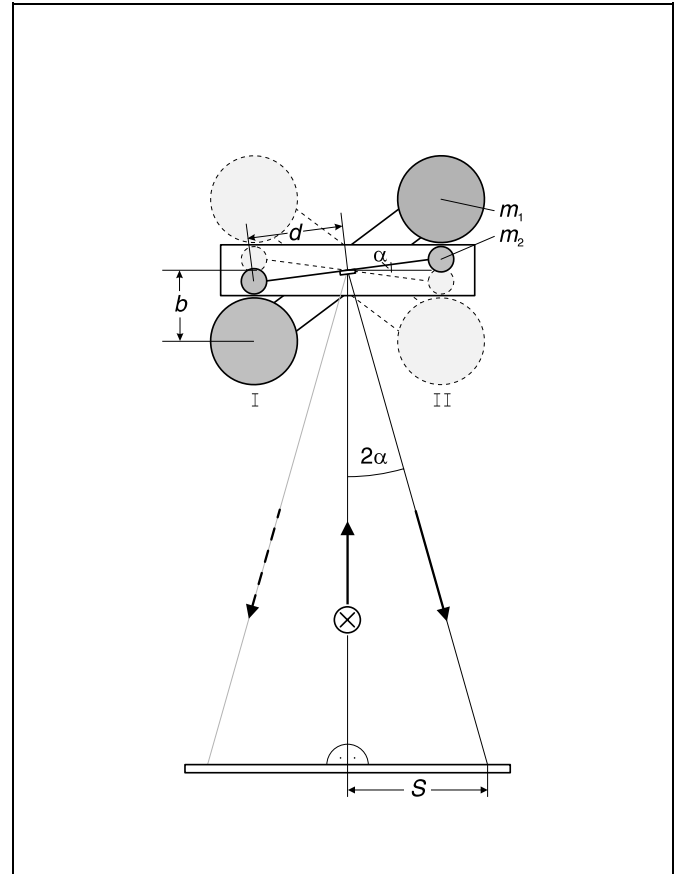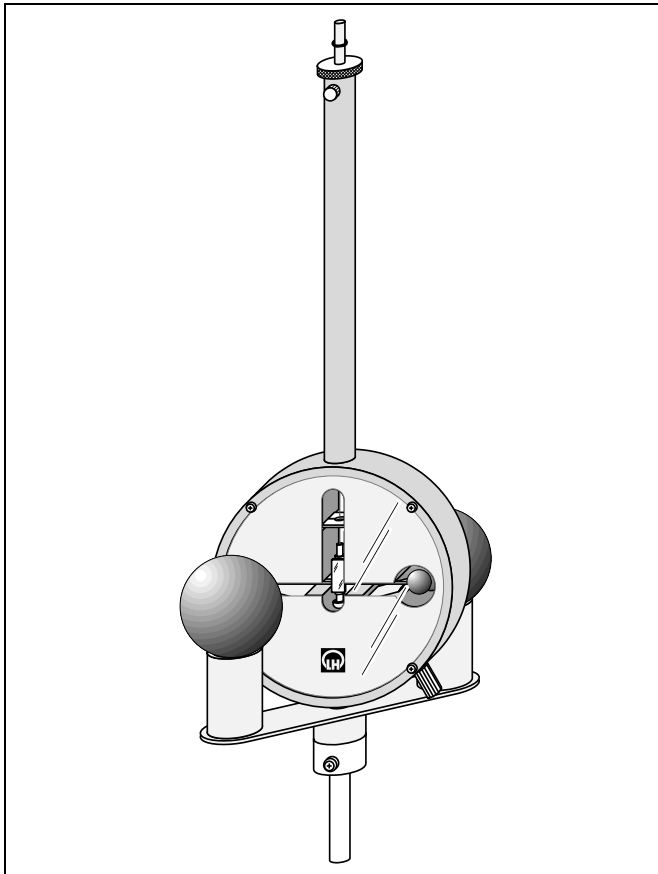
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Apparatus

End-deflection method

The end-deflection method is based on the observation that the gravitational force between two lead spheres with the masses m_1 and m_2 at a distance b is:

$$F = G \cdot \frac{m_1 m_2}{b^2} \qquad (I).$$

Thus, the moment of momentum acting on the torsion pendulum is:

$$M_I = 2\,F \cdot d = 2\,G \cdot \frac{m_1 m_2}{b^2}\,d \qquad (II)$$

when the two large lead spheres are in position I (see Fig. 1). The moment of momentum is compensated by the righting moment of the torsion cord. The torsion pendulum thus assumes the equilibrium position α_I.

By swivelling the large lead spheres in position II, the forces are symmetrically inverted. The moment of momentum acting on the bodies is now $M_{II} = -M_I$. The pendulum executes damped oscillations around the equilibrium position α_{II}. For the difference of the two moments of momentum, we can say:

$$D \cdot (\alpha_I - \alpha_{II}) = M_I - M_{II} = 2\,M_I \qquad (III).$$

The angular directional quantity D can be determined from the oscillation period T and the moment of inertia J of the torsion pendulum:

$$D = J \cdot \frac{4\pi^2}{T^2} \qquad (IV).$$

The moment of inertia J is equivalent to the moment of inertia of the two small balls

$$J = 2\,m_2 \cdot d^2 \qquad (V).$$

Thus, (IV) is transformed to

$$D = m_2 \cdot d^2 \cdot \frac{8\pi^2}{T^2} \qquad (VI).$$

From (I), (III) and (IV), we obtain

$$G = \frac{b^2 \cdot d}{m_1} \cdot \frac{2\pi^2}{T^2} \cdot (\alpha_I - \alpha_{II}) \qquad (VII).$$

Measuring the angle of rotation α:

Fig. 1 indirectly illustrates the measurement of the angle of rotation α using the light pointer: the light beam of the light pointer is perpendicular to the neutral position of the torsion pendulum (i.e. the resting position without the large lead spheres). The position of the light pointer for the neutral position coincides with the zero point of the scale. The respective relationship

$$\tan 2\alpha = \frac{S}{L_0}$$

or

$$\alpha = \frac{S}{2\,L_0} \qquad (VIII)$$

exists between the angle of rotation α, the light-pointer position S and the distance L_0 between the scale and the torsion pendulum, as angle α is very small.

In Fig. 2, the concave mirror is illuminated in the horizontal angle β. The light pointer position O for the zero position of the torsion pendulum is at a distance of L_1 from reference point N on the normal line and a distance of

$$L = \sqrt{L_0^2 + L_1^2}$$

to the concave mirror. For the rotation of the torsion pendulum by angle α from the zero position, we can find the relationship

$$S' = L \cdot \tan 2\alpha$$

and

$$\frac{S'}{S} = \frac{\sin(90° - \beta + 2\alpha)}{\sin(90° + \alpha)} = \cos\beta - \tan 2\alpha \cdot \sin\beta$$

In each case, angle α is very small (never greater than 1.5°); the dimensions of the gravitation torsion balance do not permit angles of illumination greater than 30°. Thus, the approximation

$$\frac{S'}{S} = \cos\beta = \frac{L_0}{L}$$

is permissible. With the additional approximation $2\alpha = \tan 2\alpha$, we can say, in all:

$$\alpha = \frac{S}{2} \cdot \frac{L_0}{L_0^2 + L_1^2} \qquad (IX).$$

Equation (IX) has a systematic error of 1 to 2 %; when calculating the difference of the two equilibrium positions $\alpha_I - \alpha_{II}$, this systematic error is compensated almost completely.

For the special case of illumination at the angle $\beta = 0°$, $L_1 = 0$ is to be assumed; equation (IX) then gives us the derived equation (VIII).

Equation (IX) is also valid when the illumination beam is angled upward or downward. Here too, the read-off scale is aligned horizontally, and the height changes of the light pointer are ignored.
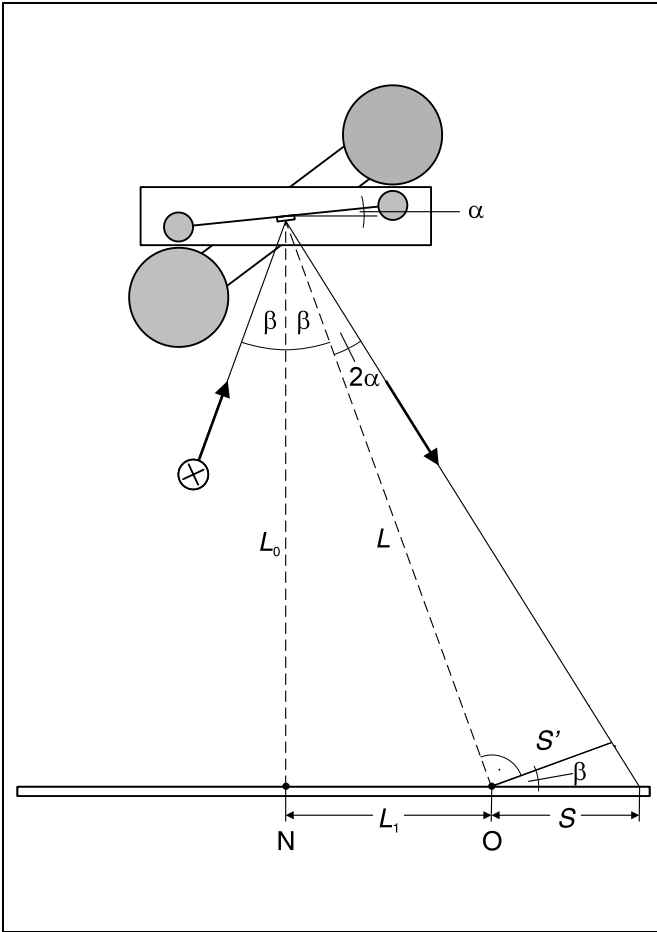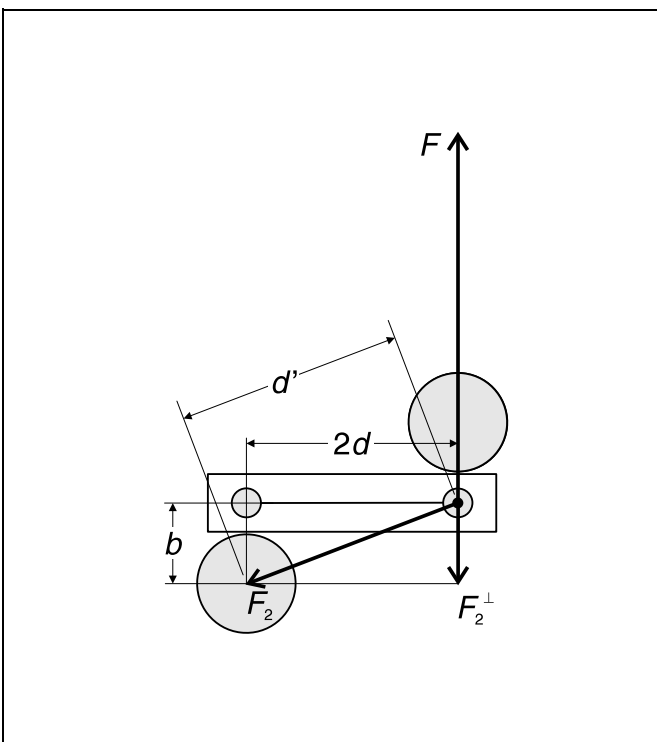
Fig. 2: Diagram of deflection measurement with the light pointer

Fig. 3: Diagram showing the calculation of the antitorque moment of the "second" sphere

The neutral position of the torsion pendulum (Point O in Fig. 2) is generally not known before the experiment is carried out. Thus, to determine L_1, measure the distance between reference point N on the normal line and the light-pointer position for equilibrium position I in close approximation. This approximation is permissible, as $\alpha << 1$. If the concave mirror is not illuminated at too steep an angle, i.e. $\beta << 1$, we can assume $L_1 = 0$.

Solution equation for the end-deflection method:

From (VII) and (IX), we derive the equation for determining the gravitational constant:

$$G = \pi^2 \cdot \frac{b^2 \cdot d}{m_1} \cdot \frac{S_I - S_{II}}{T^2} \cdot \frac{L_0}{L_0^2 + L_1^2} \tag{X}$$

The quantities $m_1 = 1.5$ kg, $d = 50$ mm and $b = 47$ mm are apparatus constants. The quantities to be measured are the oscillation period T of the torsion pendulum, the distance between the two light-pointer positions S_I and S_{II} for the equilibrium positions α_I and α_{II} and the geometry of the light pointer. The determination of S_I and S_{II} is described in the following using an example.

Antitorque moment of the "second" lead sphere:

In addition to the moment of momentum caused by the attractive force F of the respective opposing large lead spheres (distance b), the attractive force F_2 of the respectively more distant sphere (distance d') gives rise to an antitorque moment (see Fig. 3). Thus, to formulate it more precisely than equation (II), the following applies for the moment of momentum M_I:

$$M_I = 2 (F + F_2^\perp) \cdot d,$$

where

$$F_2^\perp = -F_2 \cdot \frac{b}{d'}$$

is the component of force

$$F_2 = F \cdot \frac{b^2}{d'^2}$$

which is acting perpendicular to the transverse beam (see (I)). The gravitational constant G is thus greater than calculated in (X) by the correction factor

$$K = \frac{F}{F + F_2^\perp} = \frac{1}{1 - \dfrac{b^3}{d'^3}} \tag{XI}$$

With

$$d' = \sqrt{(2d)^2 + b^2}$$

we can calculate the numerical value $K = 1.083$.

Acceleration method:

Directly after the large lead spheres are slewed from position I to position II, the twisted torsion thread exerts a righting moment $-M_I$ on the transverse beam, so that the total moment of momentum acting on the beam is $2M_{II}$. This causes the angular acceleration

$$\ddot{\alpha} = \frac{2M_{II}}{J},$$

As an approximation, we may assume that this acceleration remains constant during the first minute. Thus, during this time, we can say that

$$\alpha = \alpha_I + \frac{M_{II}}{J} \cdot t^2 \tag{XII}$$

From (II), (V) and (XII), we obtain

$$G = \frac{b^2 \cdot d}{m_1} \cdot \frac{\alpha - \alpha_I}{t^2}$$

and, from this in turn, we can use (IX) to obtain

$$G = \frac{b^2 \cdot d}{m_1} \cdot \frac{1}{2} \cdot \frac{S - S_I}{t^2} \cdot \frac{L_0}{L_0^2 + L_1^2} \tag{XIII}$$

Thus, in the acceleration method, the change $S - S_1$ of the light-pointer position is measured as a function of the time t and the geometry of the light pointer.

Setup

Important: satisfactory measuring results are only possible when the torsion balance is adjusted properly. In addition, the torsion oscillations caused by the attraction between the masses must not be disturbed by undesired pendulum motions. The torsion pendulum is extremely sensitive to shocks transmitted to the housing of the torsion balance. Changes in temperature cause convection currents in the housing of the torsion balance, which in turn results in unwanted movements of the torsion pendulum.

Choose a stable experiment setup on a solid wall.

Select an experiment site which is not exposed to direct sunlight or drafts.

When slewing the sphere support, avoid shocks to the housing e.g. by knocking it with the lead spheres.

Fig. 4 shows the experiment setup.

Assembling the torsion balance before using it the first time:

– Embed an 8-mm screw solidly in the wall. Hang the stand base by its apex hole and align it parallel to the wall using the adjusting screws (see Instruction Sheet).

– Clamp the stand rod in the base.

– Attach the gravitation torsion balance (without the large lead spheres) so that there is sufficient space to turn the supporting arm of the lead spheres.

– Align the gravitation torsion balance as nearly exactly vertical as possible, with the glass plate of the housing parallel to the wall for the scale.

– Loosen the arresting mechanism of the torsion pendulum and correct the alignment so that the pin at the end of the pendulum hangs in the middle of the rod hole, and the torsion pendulum can swing freely.

– Allow the torsion pendulum to hang for one to two days, and readjust the zero point if necessary (see Instruction sheet).

Light pointer:

– Mount the laser with rotating clamp at a distance of about 35 cm and aim it at the concave mirror. Make sure that the large lead spheres do not obstruct the path of the light pointer either in position I or position II.

– Attach the lead spheres and swivel the supporting arm to position I.

– Check the vertical alignment of the setup.

Note: at a distance of 10 m between the torsion balance and the scale, the distance between the two end positions on the scale is about 35 cm. The total range covered by the light pointer is approx. 1 m.

– Glue the scale horizontally to the intended wall or a stable base so that light-pointer position I is at about 35 cm, or light-pointer position II is at about 65 cm.

Fig. 4   Experiment setup with wall mounting
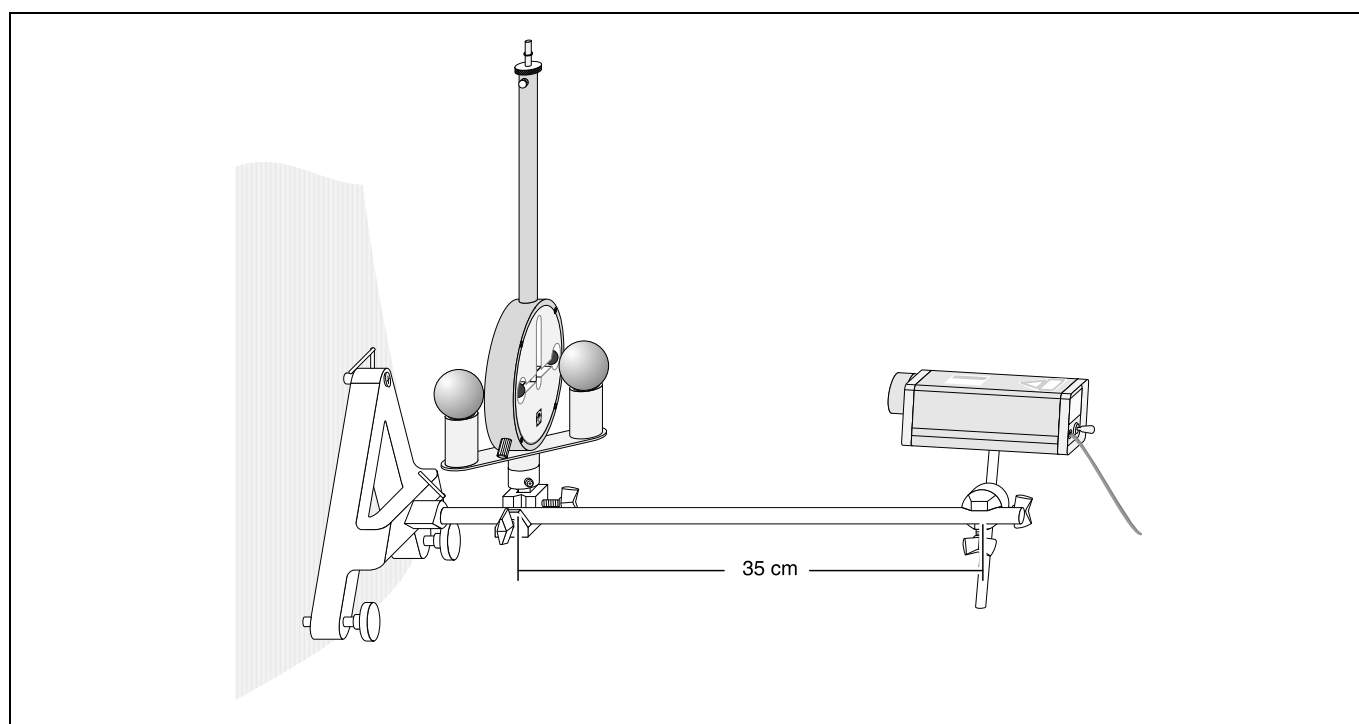
Carrying out the experiment

First:

– Allow the setup to stand for at least two hours undisturbed by shocks, so that the pendulum can come to rest in the equilibrium position (if the arresting screw has been loosened after a long period of disuse, the torsion pendulum may require more time to settle into the equilibrium position).
– Check the stability of the zero point.
– Observe the zero-point fluctuations for at least 10 minutes.
– Measure the distance L_0 from the torsion pendulum to the wall, and the distance L_1 from point N to position S_I (see Fig. 2), and write these down.

a) End-deflection method:

– Wait until the system settles into a stable equilibrium position, then find S_I and write this value down.
– Move the supporting arm with the lead spheres rapidly (but carefully!) from position I to position II and start the stop-clock.
– Read the position of the light pointer on the scale at least every 30 seconds for at least three oscillation periods, and write these values down.

Alternative method for determining S_I:

– Swivel the lead spheres from position II back to position I and repeat the measurement.

b) Acceleration method:

– Wait until the system settles into a stable equilibrium position, then find S_I and write this value down.
– Move the supporting arm with the lead spheres rapidly (but carefully!) from position I to position II and start the stop-clock.
– Read the position of the light pointer on the scale at least every 10 seconds over one oscillation period, and write these values down.

Measuring example

$L_0 = 6385$ mm, $L_1 = 1370$ mm

The zero point of the scale does not coincide with the light-pointer position for the zero position of the torsion pendulum (as assumed in Fig. 1 and Fig. 2). L_1 is the distance between light pointer position S_1 and reference point N on the normal line.

a) End-deflection method:

$S_I = 650$ mm

Table 1: End position S of the light pointer after swivelling the large lead spheres to position II

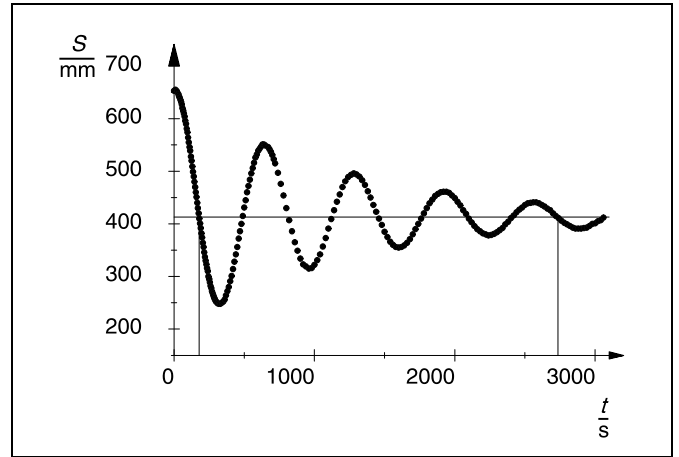| No.	$\dfrac{S}{\text{mm}}$
1	248
2	551
3	315
4	496
5	355
6	407
7	405

Fig. 5: Oscillations of the gravitation torsion balance around the final equilibrium position S_{II}

b) Acceleration method:

$S_I = 650$ mm

Table 2: Light-pointer position S in the first minute after swivelling the large lead spheres to position II

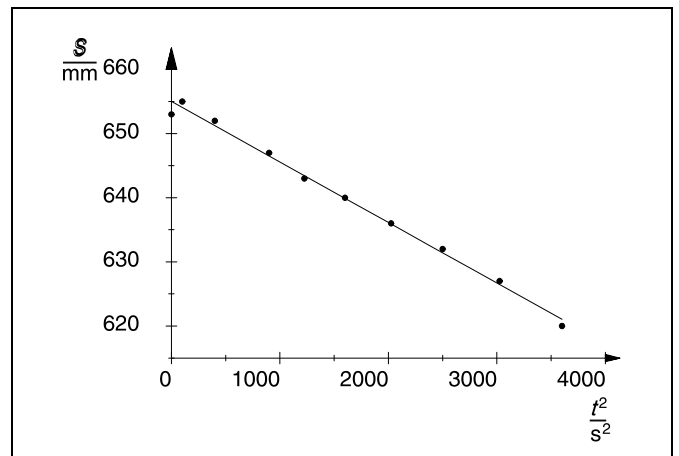| $\dfrac{t}{\text{s}}$	$\dfrac{S}{\text{mm}}$
0	653
10	655
20	652
30	647
35	643
40	640
45	636
50	632
55	627
60	620

Fig. 6: Light-pointer position S as a function of t^2 in the first minute after swivelling the large lead spheres

Evaluation and results

Apparatus constants:

$m_1 = 1.5$ kg
$d = 50$ mm
$b = 47$ mm
Correction factor $K = 1.083$

a) End-deflection method:

From four oscillation periods in Fig. 5, we can calculate $T = 639$ s.

Using the first five end-deflections $S^{(1)} - S^{(5)}$ of the light barrier (see Table 1), it is possible to calculate the equilibrium position S_{II}:

$$S_{II} = \frac{1}{2} \cdot \left(\frac{S^{(1)} + S^{(3)} + S^{(5)}}{3} + \frac{S^{(2)} + S^{(4)}}{2} \right) = 415 \text{ mm}$$

Without correction, equation (X) provides the result

$$G = 6.29 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$$

With correction, the result is

$$G = 6.81 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$$

Literature value: $G = 6.67 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$

b) Acceleration method:

Fig. 6 shows the linear relationship between position S of the light pointer and the square of time t in the first minute following the change in the position of the spheres. The slope has the value:

$$\frac{S_I - S}{t^2} = 9.4 \cdot 10^{-3} \frac{\text{mm}}{\text{s}^2}$$

Without correction, equation (XIII) gives us the result:

$$G = 5.2 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$$

With correction, the result is

$$G = 5.6 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$$

As the acceleration of the lead balls decreases over time due to the decrease in thread torsion, the best-fit straight line for finding the ratio

$$\frac{S - S_I}{t^2}$$

has a systematic error. Closer investigation shows that this error amounts to -7.5 % when one tenth of the oscillation period T, i.e. approx. 60 s is registered.

Correction for this error gives us the value

$$G = 6.0 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$$

Additional information

The distance value $b = 47$ mm is based on the assumption that the two small lead balls are centered in the housing and the large lead spheres are touching the outer wall. If our calculation of the gravitational constant in (I) takes the change in distance due to pendulum motion into consideration (the distance is 4–5 mm shorter in the equilibrium position), this changes both the result for the equilibrium positions (III) and the relationship between the angular directional quantity D and the oscillation period T (IV). An exact calculation shows that these two effects cancel each other out, so that (VII) represents a close approximation.

On the other hand, the evaluation must take any distance which may exist between the outer wall and the large lead spheres into consideration. An additional distance of 1 mm exaggerates the result for the gravitational constant by about 4 %. In comparison, the effect of inadequate adjustment of the pendulum zero point is negligible.